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SOLITARY WAVES AND THE STRUCTURES OF DISCONTINUITIES IN NON-DISSIPATIVE MODELS WITH COMPLEX DISPERSION†

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Stationary solutions of reversible evolutionary equations of mechanics with higher derivatives are analysed. A two-dimensional graphical method for investigating the solutions of systems of ordinary differential equations is described, which enables one to find special types of solutions: periodic waves, solitary waves and the structures of discontinuities. At the same time, solitary waves can be obtained by taking the limit of sequences of periodic waves and the structures of discontinuities obtained by taking the limit of sequences of solitary waves. This general approach has enabled the existence of all earlier predicted structures to be verified has enabled new types of structures (three-wave structures) to be revealed and has enabled all the necessary conditions at the discontinuities to be found. All the previously known types of solitary waves are found and new types of solitary waves are revealed (generalized ordinary and 1:1 multisolitons). Methods of finding generalized solitary waves, including those with a finite amplitude of the periodic component, are determined. Examples of the solution of the following problems are given for a fourth-order system: generalized solitary waves as the limiting solutions of two-wave resonance solutions, generalized solitary waves and the structure of a discontinuity with three waves, a 1:1 soliton and the structure of a discontinuity with a single radiated wave, a solitary wave with fixed propagation velocity, and the structure of a discontinuity in the form of a kink with radiation. A generalized 1:1 soliton and the structure of a discontinuity with two radiated waves is considered in the case of sixth-order systems. The discussion is mainly based on the example of travelling waves described by the generalized Korteweg–de Vries equations. Other models with complex dispersion (a plasma and a stratified fluid) are also considered.

A method for determining the type of structure of a discontinuity from the mutual disposition of the dispersion curve and the straight line corresponding to the phase velocity has been described for non-dissipative models [1] (by the term discontinuities, we mean any transitions between homogeneous, periodic and quasi-periodic states which do not expand with time). The main types of discontinuities are: of the soliton type, a kink and a jump with radiation. The derivative types of discontinuities are: of the generalized soliton type, a kink with radiation and a jump with two radiated waves. A similar analysis was carried out in [2] for solitary waves.

A necessary condition for a discontinuity to be stable is the evolutionary property condition: the number of boundary conditions at a discontinuity is equal to the number of emerging characteristics of the simplified system plus one. The main boundary conditions are obtained from the integral form of the initial system and the additional boundary conditions are obtained when analysing the structure of the discontinuity [3]. Discontinuities of the kink type, a jump with radiation, a kink with radiation and a jump with two radiated waves are evolutionary and the number of additional boundary conditions for these types of discontinuities are 1, 2, 3 and 4, respectively [1]. Averaged equations (for the method of deriving them, see [4, 5]) can play the role of the simplified equations for the wave zones.

A method of searching for structures was outlined in [1, 6] but it is difficult to implement. A single method for finding the structures of the above-mentioned discontinuities and the additional boundary conditions at them, which is effective when applied, is developed below. The basic idea of this method is the same the way a solitary wave is sought as the limiting solution for periodic waves and the structure of a discontinuity is sought as the limiting solution for a sequence of solitary waves. The method is based on the reversibility of non-dissipative discontinuities. A solution, which is composed of forward and reverse discontinuities, separated by a sufficiently large distance, can be considered as an ordinary or generalized solitary wave. A graphical method, which is a generalization of the method of phase portraits for second-order systems of equations, is used to find solitary waves and periodic solutions.

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1. TYPICAL NON-DISSIPATIVE EQUATIONS AND THE METHOD OF FINDING STATIONARY SOLUTIONS OF A SPECIAL TYPE

The scalar equations

$$a_t - (a^3)_x + b_3 a_{xxx} = 0 \tag{1.1}$$

$$a_t + (a^2)_x + b_3 a_{xxx} + b_5 a_{xxxxx} = 0 \tag{1.2}$$

$$a_t - (a^3)_x + b_3 a_{xxx} + b_5 a_{xxxxx} = 0 \tag{1.3}$$

are analogues of the Korteweg–de Vries (KdV) equations.

Equation (1.1) is a modified KdV equation with a cubic non-linearity and describes models with the degeneration of a quadratic non-linearity. The generalized KdV equation with a fifth-order derivative (1.2) describes the propagation of waves when there is an ice coating [7] and in a plasma [8]. The generalized KdV equation with a fifth-order derivative and a cubic non-linearity (1.3), which has the same properties as Eqs (1.1) and (1.2), is treated in this paper as a model for developing a method for obtaining a structure in the form of a kink with radiation.

The generalized non-linear Schrödinger equation with a third-order derivative

$$A_t + b_1 A_x + i b_2 A_{xx} + b_3 A_{xxx} + i|A|^2 A = 0 \tag{1.4}$$

describes an envelope of wave packets with a number close to the point of inflection of the dispersion curve and, in particular, for waves in water [9]. Changing to real variables, a and w, where $A = a \exp i\psi$, $w = \psi_x$, this complex equation is equivalent to two equations with third-order derivatives [6].

The equations for a cold plasma [8] are an example of a system of equations with many unknowns

$$\frac{\partial n}{\partial t} + \frac{\partial nu}{\partial x} = 0, \quad \frac{\partial nu}{\partial t} + \frac{\partial}{\partial x}\left(nu^2 + \frac{B_y^2 + B_z^2}{2}\right) = 0$$

$$\frac{\partial nv}{\partial t} + \frac{\partial}{\partial x}\left(nuv - B_x B_y + R_e^{-1}\frac{dB_z}{dt}\right) = 0, \quad \frac{\partial nw}{\partial t} + \frac{\partial}{\partial x}\left(nuw - B_x B_z - R_e^{-1}\frac{dB_y}{dt}\right) = 0 \tag{1.5}$$

$$\frac{\partial B_y}{\partial t} + \frac{\partial}{\partial x}\left(uB_y - B_x v - R_i^{-1}\frac{dw}{dt}\right) = 0, \quad \frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}\left(uB_z - B_x w + R_i^{-1}\frac{dv}{dt}\right) = 0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \quad R_e = \sqrt{\frac{m_i}{m_e}} \gg 1, \quad R_i = R_e^{-1} \ll 1$$

n is the ion density, $\mathbf{B} = (B_x, B_y, B_z)$ is the magnetic induction vector, $B = \text{const}$ and $\mathbf{v} = (u, v, w)$ is the velocity of the ions.

When investigating travelling waves (see below), Eq. (1.1) leads to a system of ordinary second-order differential equations and the remaining equations lead to fourth-order systems. It makes sense also to investigate equations which lead to higher-order systems where new types of discontinuities appear. Laminar and multicomponent media are a source of this kind of equations. For example, on applying the standard method for deriving the equations for long waves [10], taking account of elastic effects at the interfaces [7, 11], it is possible to derive analogues of the Boussinesq equations for a multilayer fluid with elastic plates between the layers

$$u_{it} + u_i u_{ix} + g\eta_{ix} - \frac{1}{3}H_i^2 u_{ixxt} + \frac{1}{2}H_i \eta_{i+1xtt} + \frac{1}{\rho_1}p_{ix} = 0, \quad \eta_{it} + [(H_i + \eta_i - \eta_{i+1})u_i]_x = 0$$

$$p_i = p_{i-1} + p_{hi-1} + p_{\sigma i}$$

$$p_{hi} = \rho_i[g(H_i + \eta_i - \eta_{i+1}) - 1/2 H_i^2 u_{ixt} + H_i \gamma_{i+1tt}]$$

$$p_{\sigma i} = \frac{E_i h_i^3}{12(1 - \nu_i^2)}\eta_{ixxxx} - h_i \sigma_i \eta_{ixx} + r_{\sigma i} h_i \eta_{itt}$$

where u_i are the mean velocities, p_i is the pressure under the plates, p_{hi} are the hydrostatic and hydrodynamic components, $p_{\sigma i}$ are the elastic components, η_i are the displacements of the interfaces, E_i are Young's moduli, ν_i are Poisson's ratios, H_i are the thicknesses of the fluid layers, h_i are the thicknesses of the plates, ρ_i are the densities of the fluid layers and $\rho_{\sigma i}$ are the densities of the plate materials. A plate with the number i separates the $(i-1)$th and the ith layers, $i = 1, \dots, m$. The zeroth layer is air, the $(m+1)$th layer is the bottom, $p_{h0} = 0$, p_0 and η_{m+1} are specified functions of x and t.

Additional higher non-linear and dispersion terms must be included in such equations in the general case in order that higher-order dispersion effects are taken into account asymptotically correctly. However, in the simplest case of two fluid layers with an elastic plate between them (submerged ice), this does not have to be done, due to the existence in the model of elastic parameters which depend on the hydrodynamic parameters

$$\begin{aligned}
&u_{1t} + u_1 u_{1x} + g\eta_{1x} + \frac{1}{3}gH_1^2\eta_{1xxx} + \frac{1}{2}gH_1H_2\eta_{2xxx} = 0\\
&\eta_{1t} + [(H_1 + \eta_1 - \eta_2)u_1]_x = 0\\
&u_{2t} + u_2u_{2x} + g\eta_{2x} + g\frac{\rho_1}{\rho_2}(\eta_{1x} - \eta_{2x}) + c_3\eta_{2xxx} + c_5\eta_{2xxxxx} + \frac{\rho_1}{2\rho_2}gH_1^2\eta_{1xxx} = 0 \qquad (1.6)\\
&\eta_{2t} + [(H_1 + \eta_2)u_2]_x = 0\\
&c_3 = \frac{1}{3}gH_2^2 - \frac{h\sigma}{\rho_2} + \frac{\rho_\sigma hgH_2}{\rho_2} + gH_1H_2, \quad c_5 = \frac{Eh^2}{12(1-\nu^2)\rho_2}
\end{aligned}$$

Here, to place the terms with the higher derivatives in order, the time derivatives have been replaced by spatial derivatives, which is equivalent in this approximation.

The investigation of the stationary solutions of system (1.6) leads to sixth-order system. The generalized KdV equation with a seventh-order derivative

$$a_t + aa_x + b_3a_{xxx} + b_5a_{xxxxx} + b_7a_{xxxxxxx} = 0 \qquad (1.7)$$

is considered as the model equation for developing methods for investigating such systems in this paper.

In the equations described above, we change to a system of coordinates moving with a velocity V, we equate the time derivative to zero and investigate the stationary solutions of the equations being considered. The initial equations can be represented in the form of conservation laws and, therefore, after a single integration of the system of travelling wave equations, we obtain a system of equations with an even number of unknowns

$$d\mathbf{u}/dx = \mathbf{F}(\mathbf{u}) \qquad (1.8)$$

This system is invariant under the transformation $x \to -x$, $u_{2p} \to u_{2p}$ (symmetrical unknowns) and $u_{2p+1} \to u_{2p+1}$ (antisymmetrical unknowns). The system depends on a parameter of the phase velocity V. One equilibrium state is considers of system (1.8) is linearized with respect to it

$$d\mathbf{u}'/dx = A\mathbf{u}', \quad \mathbf{u} = \mathbf{u}_h + \mathbf{u}'$$

The solutions of the linear system are found analytically and can be used to choose the initial conditions. It is assumed that the solutions of non-linear system (1.8), with certain initial data, are easily obtained numerically. Solutions are sought which are symmetrical with respect to x about a certain point $x = x_s$ and which, when $x \to \pm\infty$, possess the desired properties (periodicity, tending to an equilibrium state). In order to do this, a certain set of phase trajectories S_1 (an invariant manifold) is considered in phase space which is constructed using a certain set of initial data S_0, which provides the required property when $x \to -\infty$, for example, the approach to a certain equilibrium point or to a certain cycle. A symmetrical solution of the required type (periodic or soliton) corresponds to each point of intersection of the manifold S_1 with the plane S, where all but one of the antisymmetrical variables are equal to zero. The discontinuities in the models being considered are reversible, and a solution in the form of two mutually reversible discontinuities (the two discontinuities are not always evolutionary) separated by a large distance can be considered as a solitary or a generalized solitary wave.

Methods of finding solutions of a different type will be described below. We adopt the following general notation: ε is a small parameter associated with the use of linearized solutions, C or const is a finite constant quantity in the calculation, var is a quantity which varies in a calculation with a certain discrete step size, $\varkappa$ is characteristic number ($\varkappa = ik$, where k is a wave number and the values of $\varkappa$ are matched with the conditions for the structures of the discontinuities to exist [1]) and $\mathbf{u}_\varkappa$ is an eigenvector of the matrix A. Only half the characteristic numbers are shown since their symmetry holds: if $\varkappa$ is a characteristic number then $-\varkappa$ is also a characteristic number. Apart from what has been explicitly stated, the magnitude of x is an implicit varying parameter every where (which does not require special discretization during the programming, since it is automatically provided when calculating (1.8)). The invariance of the system with respect to a phase shift is used here.

A two-dimensional projection onto the (u_0, u_3) plane of the section S_1 by the plane S is considered for the fourth-order equations: $u_1 = 0$ (or (u_0, u_1) when S: $u_3 = 0$). There is one explicit varying parameter. It can be included in the number of spatial variables and the corresponding projections can be considered. In the general case, a projection is a branching curve with a denumerable number of branches and a solution of the type being considered corresponds to each point of intersection of this curve with the axis of the non-antisymmetrical variable. On account of the complex structure of a projection, it is convenient to consider point graphs of it rather than graphs in the form of connected curves. In this case, only the main branches will be seen. The minor branches will be thinned out or not seen at all. They can be successively investigated by reducing the discretization step size and contracting the domain of variation at the same time.

Typical problems for fourth-order equations are described in Sections 2–6. A projection on to the (u_0, u_3) plane when S: $u_1 = 0, u_5 = 0$ is considered in the case of the sixth-order equations. Here, there are two explicitly varying parameters. In the case of these equations, it is also possible to solve the problems described in Section 2–6 but the specific problems described in Sections 7 and 8 also appear. The generalization to system of higher order is obvious. The problems are entitled according to the principle: a general solution or a special limiting solution.

2. PERIODIC WAVES – SOLITARY WAVES

The eigenvalues are

$$\kappa_1 = iq_1, \quad q_1 \in R; \quad \kappa_2 \in R \quad \text{or} \quad \kappa_2 = iq_2, \quad q_2 \in R$$

The initial data when $x = 0$ are

$$u_0 = C, \quad u_1 = 0, \quad u_2 = \text{var} \in [u_{21}, u_{22}], \quad u_3 = 0 \ (u_2 = \text{const}, u_0 = \text{var})$$

In the case when $\varkappa_2 \in R$, branching does not occur and the pattern is the same as in the case of a second-order system: as the period increases, the solution starts to tend to a sequence of solitary waves and, at the same time, the phase trajectory tends to one of the equilibrium points.

If, however, $\varkappa_2$ is an imaginary number, the projection branches and the solutions being considered are two-wave, resonance solutions with a rational ratio of the periods. The number of visually observed periods in these solutions can be as large as desired. By taking the limit at which the ratio of the wavelengths tends to infinity, it is possible to obtain a generalized solitary wave which is a combination of a solitary wave and a periodic wave. The method of separating out the single wave solutions is to stop the calculation of Eq. (1.8) if $u_1 = 0$.

The character of all possible bounded solutions in this case has been analysed earlier by another method [12, 13]. According to these results, there is a feathery domain (a resonance tree) in the space of the initial variables for which the solution is bounded. In order to avoid cycling of the calculation, the maximum value of x has to be restricted.

An example of a projection and the characteristic types of solutions for Eq. (1.2) are shown in Fig. 1 when $b_3 = b_5 = 1$, $V = 0.1$, $C = 0.1$ ($u_i = \partial^i a/\partial x^i$). The projection of the section S_1 by the plane S: $u_1 = 0$ onto the plane (u_0, u_3) is shown. The extreme right-hand point of the bunch of points of intersection with the u_0 axis (point A in Fig. 1) corresponds to the hump of a generalized solitary wave. In the lower part on the right, the corresponding sequence of two-wave resonance solutions is shown. These solutions converge to a generalized solitary wave in such a manner that the maximum of the soliton part coincides with the maximum of the periodic part (an in-phase solitary wave). A further sequence is shown in the lower part of the figure on the left, which converges to an anti-phase generalized solitary wave. The ratio of the periods of the waves is indicated in Fig. 1. Sequences of solutions can also be separated out which converge to the generalized multisolitons from Section 5 and, with a corresponding choice of the value of V and $C \to 0$, to the multisolitons from Section 4. The extreme left point of intersection (point B in Fig. 1) corresponds to single wave solution (the curve L_0 in Fig. 1 is a single wave branch). The limiting nature of points A and B provides a simple method for finding

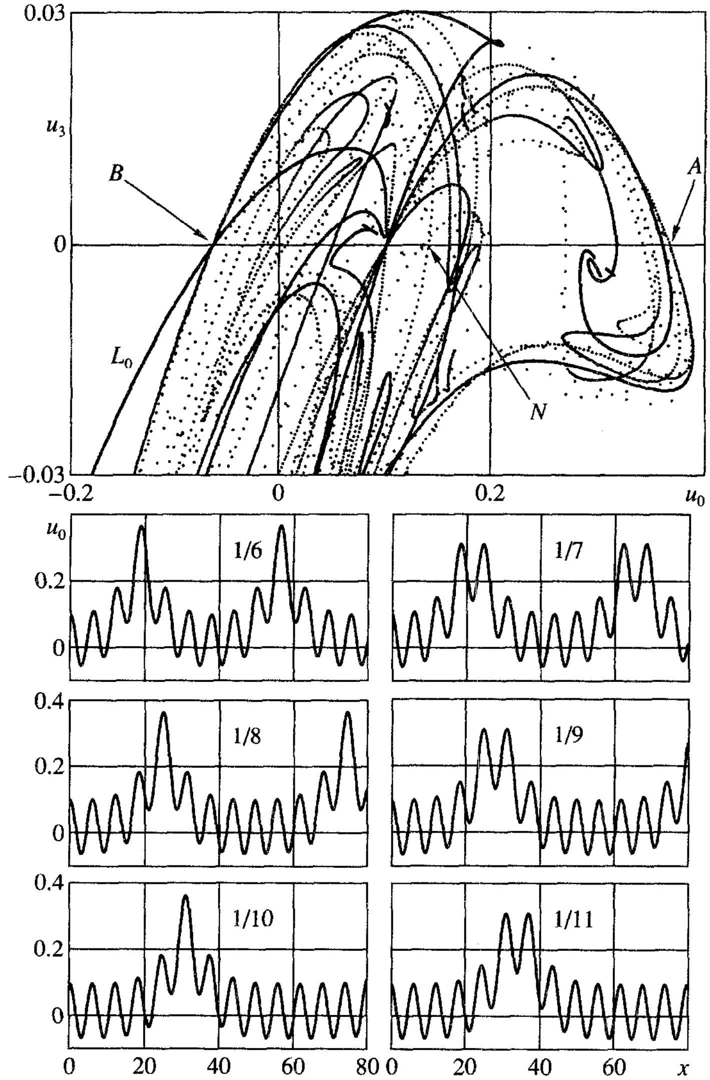

Fig. 1

the corresponding solutions. The point N corresponds to the axis of symmetry of an anti-phase generalized solitary wave.

3. CLASSICAL SINGLE-HUMP SOLITARY WAVES – KINKS

The eigenvalues are

$$\kappa_1 > 0, \quad \kappa_2 > 0$$

and the initial data are

$$\mathbf{u}(0) = \mathbf{u}_h + \varepsilon(\mathbf{u}_{\kappa 1}\sin\varphi + \mathbf{u}_{\kappa 2}\cos\varphi), \quad \varphi = \text{var} \in [0, 2\pi]$$

Kinks can be obtained from solitary waves by taking the limit when the phase trajectory tends to two different points of equilibrium and, here, the graph of such a solution is a solitary wave with a flattened vertex. This method of obtaining a solution in the form of a kink in the case of Eq. (1.1), which leads to a second-order system with a cubic non-linearity, has been demonstrated previously [4]. The phase portrait was analysed and the value of the velocity V was chosen when the separatrix passes through two equilibrium points. In this case, there is a further third equilibrium point.

Note that, in the case of second-order systems, the technique which is employed here of selecting the initial data enables one to simplify the problem and to construct the separatrix, that is, to find a solution of the solitary-wave type directly without constructing the whole of the phase portrait. In the case of fourth-order equations, the projection does not branch in this case and the process of finding a kink is therefore accomplished in a similar manner. Equation (1.4) [6] can be mentioned as an example which leads to a fourth-order system which has a solution in the form of kinds.

4. MULTIPLE-HUMP SOLITARY WAVES – STRUCTURES WITH A SINGLE RADIATED WAVE

The eigenvalues are

$$\kappa_{1,2} = r \pm iq, \quad r > 0, \quad q \in R$$

and the initial data are

$$\mathbf{u}(0) = \mathbf{u}_h + \varepsilon \mathrm{Re}(\mathbf{u}_\kappa \exp i\varphi), \quad \varphi = \mathrm{var} \in [0, 2\pi]$$

This problem has been solved in [5] in the case of Eq. (1.2). The results of a similar investigation for the system of equations of a cold plasma are presented below with the aim of demonstrating this approach. In this section the method of investigation is considered in detail and analogous techniques are used in Sections 5–7.

The dispersion curve for the linearized version of system (1.5) contains a magnetosonic and an Alfvén branch. The quantity $\theta = \mathrm{arctg}(B_x/B_{yh})$ is an important parameter of the problem [14]. When $\theta > \theta_c$, there is no point of inflection when $k > 0$ in the case of the magnetosonic branch, and, when $\theta < \theta_c$, there is such a point, where θ_c is a certain critical value. Solutions associated with the magnetosonic branch are considered. The case of four complex-conjugate values of k is possible both when $\theta > \theta_c$ and when $\theta < \theta_c$.

The stationary solutions are described by the system

$$\begin{aligned}
dv/dx &= (-B_zV + B_zu - B_zw)R_i(V-u)^{-1} \\
dw/dx &= -(-B_yV + B_yu - B_xV - c_{B_y})R_i(V-u)^{-1} \\
dB_y/dx &= -(-nwV + nwu - B_xB_z)R_e(B-u)^{-1} \\
dB_z/dx &= (-nvV + nvu - B_xB_y - c_v)R_i(V-u)^{-1}
\end{aligned} \tag{4.1}$$

$$u = n^{-1}c_n + V, \quad n^{-1} = \left(c_u - Vc_n - \frac{B_y^2 + B_z^2}{2}\right)c_n^{-2}; \quad c_n = -n_hV + n_hu_h, \quad c_u = -n_hu_hV + \frac{B_{yh}^2}{2}$$

$$c_{B_y} = -B_{uh} + u_hB_{yh} - B_xv_h, \quad c_v = -n_hv_hV + n_hu_hv_h - B_xB_{yh}$$

After linearization, system (4.1) can be written in the form

$$A(v', w', B_y', B_z')^\tau = 0$$

$$A = \begin{Vmatrix}
-n_hV + n_hu_h & 0 & -B_x & R_e^{-1}(u_h - V)\dfrac{d}{dx} \\
0 & -n_hV + n_hu_h & R_e^{-1}(V - u_h)\dfrac{d}{dx} & -B_x \\
-B_x & R_i^{-1}(V-u_h)\dfrac{d}{dx} - V + u_h + \dfrac{B_{yh}^2}{n_h(V-u_h)} & 0 \\
R_i^{-1}(u_h - V)\dfrac{d}{dx} & -B_x & 0 & -V + u_h
\end{Vmatrix}$$

$$v' = v - v_h, \quad w' = w, \quad B_y' = B_y - B_{yh}, \quad B_z' = B_z$$

We will now find particular solution of this system of the form

$$(v', w', B_y', B_z') = (1, w_k, B_{yk}, B_{zk})\exp \kappa x, \quad \kappa = ik$$

using standard algebraic methods. The value of $\varkappa$ is found from the equation

$$R(V, \kappa) = B_x^4 + B_x^2 B_{yh}^2 - 2B_x^2 n_h V'^2 - B_{yh}^2 n_h V'^2 + B_x^2 \kappa^2 R_e^{-2} V'^2 + B_{yh}^2 \kappa^2 R_e^{-1} R_i^{-1} V'^2 +$$
$$+ B_x^2 \kappa^2 R_i^{-2} V'^2 + n_h^2 V'^4 - 2\kappa^2 n_h R_e^{-1} R_i^{-1} V'^4 + \kappa^4 R_e^{-2} R_i^{-2} V'^4 = 0, \quad V' = V - u_h$$

The root with Re $\varkappa > 0$ is taken and the corresponding complex solutions $w_{\varkappa k}$, $B_{y\varkappa}$, $B_{z\varkappa}$ are found. The set of trajectories with the initial data

$$v = v_h + \varepsilon\cos\varphi, \quad w = \varepsilon \mathrm{Re}(w_\kappa z), \quad B_y = B_{yh} + \varepsilon \mathrm{Re}(B_{y\kappa} z), \quad B_z = \varepsilon \mathrm{Re}(B_{z\kappa} z) \tag{4.2}$$
$$z = \cos\varphi + i\sin\varphi, \quad 0 \le \varphi \le 2\pi$$

is considered as the numerical model of the manifold S_1. Here ε is small parameter which characterizes the accuracy of the approximation.

In order to find a solitary wave, it is necessary to take a trajectory with the initial data (4.2) and a fairly small value of ε. By varying φ and x, we find the trajectory for a certain range of values $x \in [0, x_l]$ such that $w(x_1) = 0, B_z(x_l) = 0$. For $x \in (-\infty, 0]$, the solution is proportional to exp ikx, Im$k < 0$. When $x > x_l$, the solution can be found in the following way

$$n, u, v, B_y(x_l + d) + n, u, v, B_y(x_l - d); \quad w, B_z(x_l + d) = -w, -B_z(x_l - d)$$

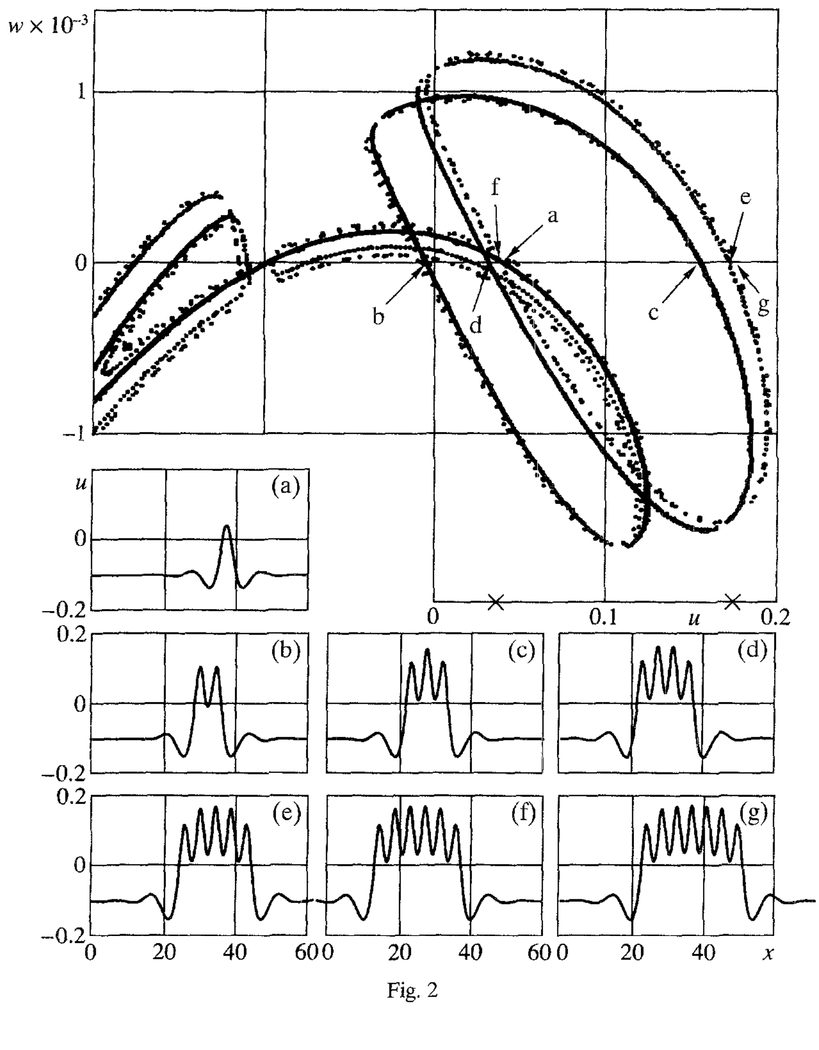

Fig. 2

A geometric approach provides a visual method of obtaining solutions of the solitary-wave type. We vary the parameter φ in the range $[0, 2\pi]$ with a certain discrete step size $\Delta\varphi = 2\pi/m$. We obtain a set of m trajectories. Suppose M is the set of points in the set of trajectories being considered, where $B_z = 0$. We now consider the projection of M onto the subspace (u, w). An example of such a projection is shown in Fig. 2, $\theta = 1.535 < \theta_c$ ($\mathrm{Re}^{-1} = 0.0231$, a hydrogen plasma).

A numerical experiment was carried out in such a way that only the domain of points which are not too far removed from the initial point of equilibrium was considered. If $m \to \infty$, then this set tends to a certain curve in the u, w plane and this curve has a denumerable number of branches. In the case of finite m, it is only possible visually to determine a finite number of the simplest branches. Each intersection of the branch with the u axis corresponds to a solitary wave. Several solitary waves are shown in Fig. 2. There is a so-called 1:1 soliton (a basic soliton) in Fig. 2(a). All the remaining solitary waves are multisolitons and combinations of basic solitons, which have been shifted by certain distances relative to one another.

The maximum and minimum values of u for a periodic state beyond a discontinuity with radiation are shown by the small crosses in Fig. 2. They can be found without analysing the structure of the discontinuity if an additional conservation law (the law of conservation of energy) is used. There is a sequence of pairs of points of intersection which tends to the points $(u_{\min}, 0)$, $(u_{\max}, 0)$. The solitary waves corresponding to these sequences of points of intersection can be considered as a sequence of solutions which tend to a combination of an evolutionary discontinuity with radiation and a discontinuity with absorption, which is the reverse of it. The first six solitary waves from this sequence are shown in Fig. 2(b–g). The corresponding points of interaction have also been marked in Fig. 2. As in the case of the generalized KdV equation, there is a denumerable number of such sequences of solitary waves and, correspondingly, of structures of discontinuities. A structure of the simplest type is realized in a transient numerical experiment. We recall that, for the given type of discontinuity, two additional boundary conditions can also be found without analysing the structure by using an additional conservation law [5], that is, these boundary conditions do not depend on the structure and, consequently, non-uniqueness is not fundamental.

In the case when the model admits of two-solutions (for a plasma when $\theta < \theta_c$), a 1:1 soliton can be sought using the method from Section 2 as the limiting solution of a sequence of two-wave solutions in which the ratio of the wavelengths tends to unity.

5. GENERALIZED SOLITARY WAVES – THREE-WAVE STRUCTURES

The eigenvalues are

$$\kappa_1 \in R, \quad \kappa_2 = iq, \quad q \in R$$

and the initial data are:

for a periodic component of small amplitude

$$\mathbf{u}(0) = \mathbf{u}_h + \varepsilon\mathbf{u}_{\kappa 1} + C\mathrm{Re}(\mathbf{u}_{\kappa 2}\exp i\varphi), \quad \varphi = \mathrm{var} \in [0, 2\pi], \quad C = \mathrm{const}$$

for a periodic component of finite amplitude

$$\mathbf{u}(0) = l\varepsilon\mathbf{u}'(0) + \mathbf{u}_p(0), \quad l = \mathrm{var} \in [l_1, 1], \quad 0 < l_1 < 1$$

The magnitude of l_1 is determined when solving the linearized system of equations

$$d\mathbf{u}'/dx = A(x)\mathbf{u}', \quad \mathbf{u} = \mathbf{u}_p(x) + \mathbf{u}', \quad \mathbf{u}'(0) = l_1\mathbf{u}'(-\Lambda) \tag{5.1}$$

The linearization is carried out with respect to the periodic component $\mathbf{u}_p$ with period Λ. This is a single-parameter family of solutions (ignoring the phase shift) which depends on the amplitude parameter C. The method of finding the single-wave solution has been indicated in Section 2.

In order to solve problem (5.1), we find the basis $\{\mathbf{u}_j'\} : u_{ij}'(0) = \delta_{ij}$ in the space of linearized solutions. Suppose $\{X_k\}$ are the coordinates of the required solution in this basis. Then, l_1 is a real eigenvalue of the matrix

$$B = (\mathbf{u}_1'(-\Lambda), \ldots, \mathbf{u}_n'(-\Lambda)), \quad B\mathbf{X} = l_1\mathbf{X}$$

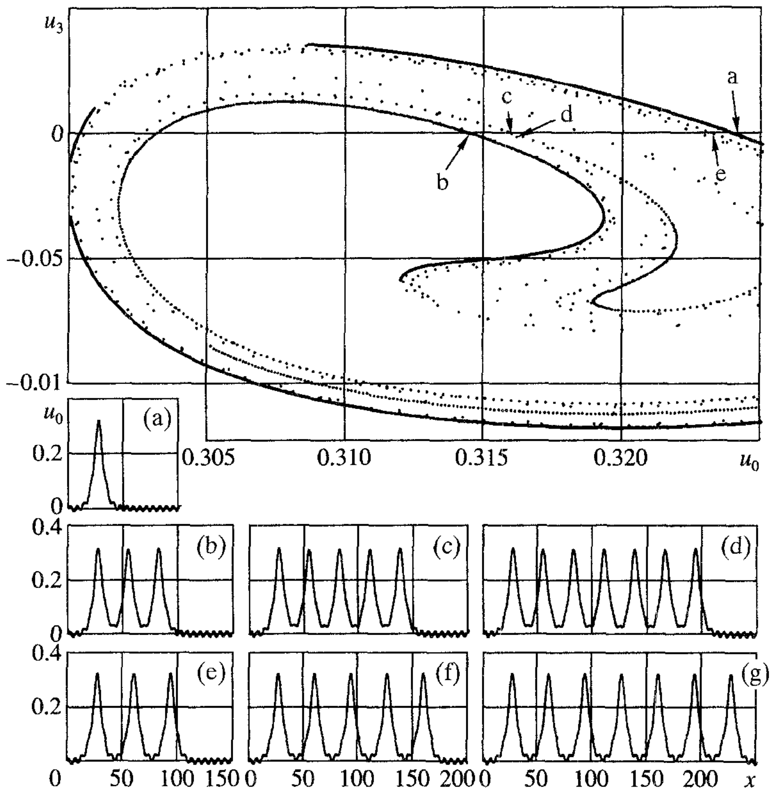

Fig. 3

For the periodic component of small amplitude in the limit

$$\mathbf{u}'(0) = \mathbf{u}_{\kappa 1}, \quad \mathbf{u}_p(0) = \mathbf{u}_h + C\mathrm{Re}\,\mathbf{u}_{\kappa 2}, \quad l_1 = \exp(-2\kappa_1\pi/q)$$

Consequently, in the case of the characteristic polynomial $P(\lambda) = \det(B - \lambda E)$, there is a real root, at least for a sufficiently small amplitude, and an solution of problem (5.1) exists.

A fragment of the projection in the case of a small amplitude of the periodic component for Eq. (1.2) when $b_3 = b_5 = 1$, $V = 0.1$, $C = 0.01$ and the sequences of generalized solitary waves a–d and a, d–g, which converge to two different three-wave structures, are shown in Fig. 3. The far right intersection (intersection a in Fig. 3) corresponds to a solution in the form of a combination of one solitary wave and a periodic wave (a generalized solitary wave). The other intersections correspond to combinations of several solitary waves, with a certain distance between them, and a periodic component (generalized multisolitons).

We will confine the treatment to an analysis of the points of intersection with the u_0 axis which are located to the right, that is, between the points a and b in Fig. 3. The intersections correspond to in-phase multisolitons and the intersection on the left corresponds to anti-phase multisolitons. Three-soliton branches are the most notable and the distance between the solitons increases in proportion to how the point of intersection, which corresponds to the solution being considered, moves along the u_0 axis from the extreme left-hand position (point b) to the extreme right-hand position (point a).

The branches corresponding to solutions with a large odd number of solitons, the distance between which is close to a certain fixed value for a given sequence, touch each three-soliton branch. Intersections, corresponding to even combinations, also exist but they are located in another part of the plane and are not shown in Fig. 3. Hence, it is possible to separate out sequences of solutions which converge to solutions in the form of two mutually inverse discontinuities, which are separated from one another by a large distance. On one side of the discontinuity there is a periodic solution and, on the other side, there is a biperiodic solution. There is a denumerable number of such structures for a given value of

V. There is a structure with the smallest distance between the solitons (a discontinuity of maximum compression) and structures with solitons which are as far apart from one another as may be desired. A discontinuity of the generalized soliton type, that is, a transition from a periodic state to a sequence of generalized solitary waves is the formal limit of a sequence of such structures. In addition to multisolitons with a regular arrangement of the soliton elements, there are various irregular multisolitons.

If the amplitude of the periodic component starts to be varied by reducing it, then a situation arises for each of the branches being considered when it begins to touch the u_0 axis. In this case, the corresponding in-phase and anti-phase solutions are identical. When the amplitude is reduced further, a solutions of this type disappears. Finally, the solution in the form of an isolated generalized solitary wave vanishes.

On the other hand, when the amplitude of the periodic component is increased, all the new branches begin to intersect the u_0 axis. Starting from a certain value of the amplitude, the three-soliton solution of maximum compression appears. This does not have a continuation in the form of regular solutions with a large number of soliton elements. However, at the same time there are irregular solutions with a small average distance between the solitons.

Unlike all the remaining structures considered in this paper, that is, theoretically predicted structures, evolutionary structures and structures which are observed in a transient numerical experiment [1, 4, 6, 11, 14], the question of the stability of the three-wave structures which have been found here is not completely clear. The use of the concept of an evolutionary character is difficult here on account of the implicit nature of the averaged equations for the two-wave zones. The existence of three-wave structures in the case of fourth-order equations was not predicted [1], since the existence of such structures is attributable to non-linear resonance effects. In this case, a three-wave discontinuity with a structure close to a stochastic structure is observed in a transient numerical experiment. It is not completely clear whether this is actually so or whether it is associated with an insufficiently prolonged observation time, scheme effects, or computational rounding errors. If it is assumed that a discontinuity of maximum compression must be achieved, then, guided by the investigations which have been carried out here, it can be foreseen that a regular solution must be formed in the case of small amplitude discontinuities with time and a stochastic solution in the case of large amplitude discontinuities.

6. SOLITARY WAVES WITH FIXED VELOCITIES – KINKS WITH RADIATION

The eigenvalues are

$$\kappa_1 > 0, \quad \kappa_2 = iq, \quad q \in R$$

and the initial data are

$$\mathbf{u}(0) = \mathbf{u}_h + \varepsilon \mathbf{u}_{\kappa 1}, \quad V = \text{var} \in [V_1, V_2]$$

The process for finding the structures in the form of a kink with radiation in the case of Eq. (1.3) $b_3 = b_5 = 1$, $V_0 = 2.618$ is illustrated in Fig. 4. Here, it is convenient to take the velocity V as the coordinate. A section and a sequence of solitary waves – multisolitons, which converges to a solution in the form of two mutually inverse kinks with radiation are shown. The structure of the projection is simpler here than in Section 5. The curve does not branch and only intersects the V axis a denumerable number of times. Point A corresponds to a single soliton solution and point B is the point where the points of intersection crowd together. All the points of intersection in Fig. 4, apart from point A, have practically merged, which means that a two-soliton solution is now sufficiently close to the structure of a kink with radiation. The solutions which are considered in this section are a special case of the solutions considered in Section 5 and could be sought using the method described there.

In order to describe it qualitatively, it is not obligatory to seek the structure of the discontinuity itself here. The use of the velocity as the coordinate enables one easily to find the additional boundary condition at the discontinuity $V = V_*$, where V_* corresponds to point B. As previously [5], it is possible to find the parameters of the homogeneous wave state using the additional conservation law.

The solitary waves being considered exist when there is an intersection of the line $V = \omega/k$ and the dispersion curve, but an additional condition is imposed on those solutions. Note that solitary waves with fixed velocities were earlier found for Eq. (1.4) [9] which, like Eq. (1.3), possesses complex dispersion and non-linear properties.

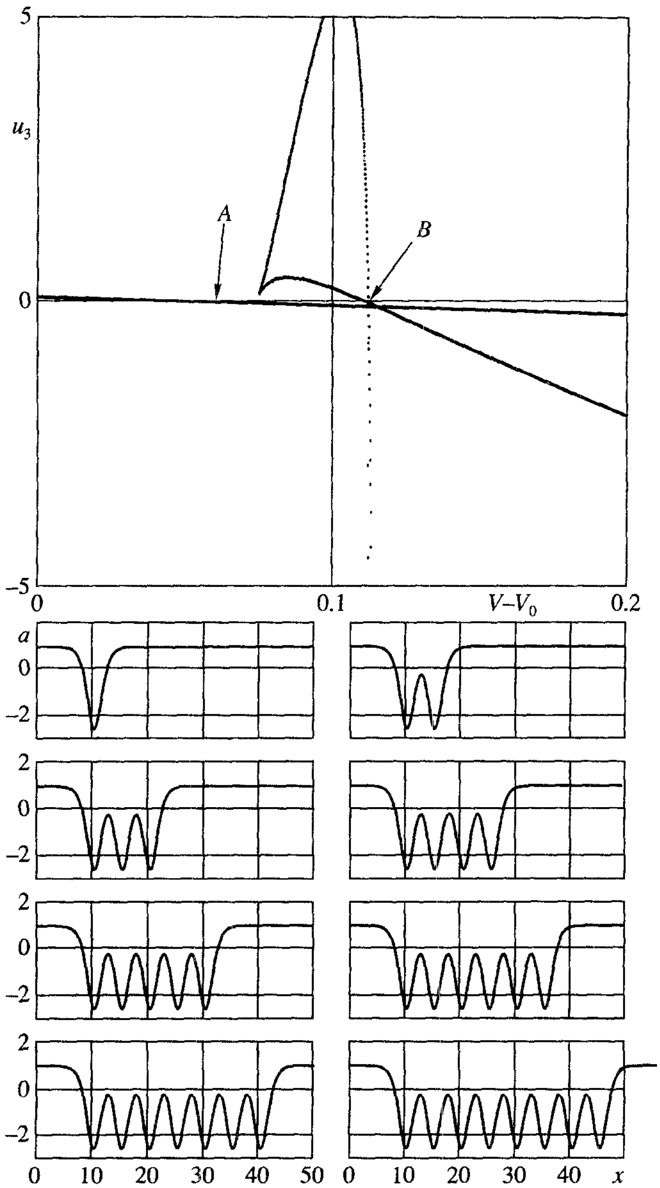

Fig. 4

7. GENERALIZED MULTIHUMP SOLITARY WAVES (1:1 SOLITONS) – THREE-WAVE STRUCTURES WITH TWO RADIATED WAVES

The eigenvalues are

$$\kappa_{1,2} = r \pm iq_1, \quad \kappa_3 = iq_2, \quad r > 0, \quad q_1 \in R, \quad q_2 \in R$$

and the initial data for a periodic component of small amplitude are

$$\mathbf{u}(0) = \mathbf{u}_h + \mathrm{Re}(\varepsilon \mathbf{u}_{\kappa 1} \exp i\varphi + C\mathbf{u}_{\kappa 3} \exp i\psi); \quad \varphi, \psi = \mathrm{var} \in [0, 2\pi]$$

A linearization with respect to the periodic solutions

$$\mathbf{u}(x_0) = \varepsilon \mathrm{Re}[\mathbf{u}'(x_0) \exp i\varphi] + \mathbf{u}_p(x_0), \quad \mathbf{u}'(x_0 - \Lambda) = \mu \mathbf{u}'(x_0)$$

$$x_0 = \mathrm{var} \in [0, \Lambda], \quad \varphi = \mathrm{var} \in [0, 2\pi]; \quad |\mu| < 1; \quad |\mathbf{u}'(x_0)| = 1$$

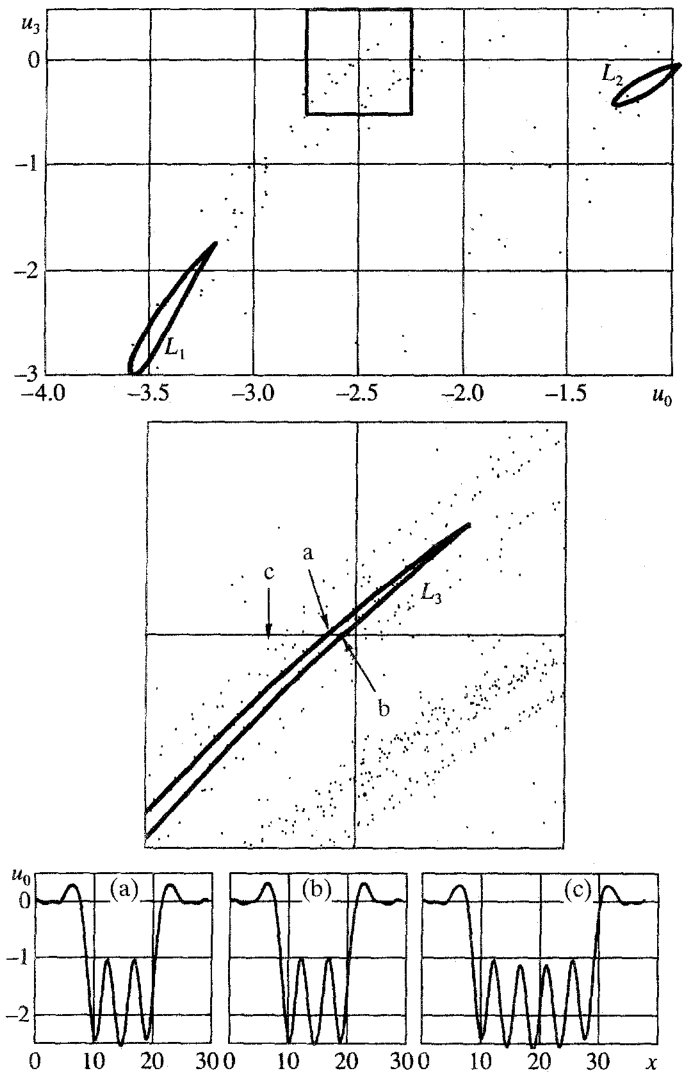

Fig. 5

is carried out in order to find the initial data in the case of a periodic component of finite amplitude.

The method of finding and substantiating the existence of the complex value of μ and the corresponding solution $\mathbf{u}'(x)$ is the same as in Section 5.

Figure 5 illustrates the process of obtaining a structure with two radiated waves for Eq. (1.7) when $b_3 = b_5 = 1, b_7 = 0.15, V = -1, C = 0.025$ in the case when the amplitude of one of the waves is small. A fragment of the projection is shown in the top diagram of Fig. 5 and the domain enclosed in the frame is shown below on a larger scale and with a reduction in the discretization step of the varied parameters by a factor of 20. Three characteristic solutions are shown in Fig. 5 and the points, from which these solutions are constructed, are indicated by arrows. This problem is actually a generalization of the problems described in Sections 4 and 5 and the nature of the projection and of the corresponding solutions has common features with both of these problems simultaneously.

There are solutions in the form of generalized solitary waves – multisolitons. They can easily be imagined if an additional periodic component is superimposed on the solutions from Section 4. As in Section 5, there are in-phase and anti-phase sequences of generalized multisolitons, which tend to the corresponding solutions in the form of forward and reverse three-wave discontinuities. A solution in the form of a discontinuity with radiated waves is obviously to be sought among them as the solution with the minimum amplitude of a radiated wave. The process is similar to the search for the generalized solitary wave with the minimum amplitude of the periodic component (see Section 5). For an arbitrary value of this amplitude, there are sequences which converge to three-wave structures. As the amplitude decreases, solutions with two, three, etc. soliton components vanish such that, on reaching a certain minimum amplitude, the solution with an infinite number of soliton components remains, and this solution represents two mutually opposite discontinuities between two periodic components.

Thus, a closed curve L_1 (the curves considered here are enclosed by a thick line) is seen in Fig. 5, corresponding to a combination of a single 1:1 soliton and a periodic wave. The curve L_2 corresponds to a two-soliton solution and it almost touches the u_0 axis, whereas L_1 is separated from it. For large values of the amplitude of the periodic component, these curves intersect the u_0 axis. When the resolution is increased, the curve L_3 becomes visible. This curve corresponds to a three-soliton solution and intersects the u_0 axis twice. The corresponding in-phase (Fig. 5a) and anti-phase (Fig. 5b) generalized solitary waves are almost identical. It is obvious that this intersection vanishes when the amplitude is reduced further. Five-soliton solutions are recognized here (Fig. 5c) but the resolution is insufficient to treat the curve L_5 in Fig. 5 in detail. Unlike in the case of the fourth-order equations, where the variation of a single parameter and the choice of solutions which converge to the structures of the discontinuities is easily accomplished and without a subsequent geometrical procedure, the method of a step-by-step increase in the resolution turns out to uniquely possible in the case of sixth-order equations.

As in the case of a kink radiation, a single additional conservation law is found to be insufficient to obtain all the necessary additional boundary conditions. Here, a complete investigation of the structure is required in order to find all the necessary additional boundary conditions.

8. GENERALIZED SOLITARY WAVES WITH FIXED VELOCITIES – STRUCTURES WITH TWO RADIATED WAVES

The eigenvalues are

$$\kappa_1 \in R, \quad \kappa_2 = iq_1, \quad \kappa_3 = iq_2, \quad q_1 \in R, \quad q_2 \in R$$

and the initial data are

$$\mathbf{u}(0) = l\varepsilon\mathbf{u}'(0) + \mathbf{u}_p(0), \quad l = \text{var} \in [l_1, 1], \quad V = \text{var} \in [V_1, V_2]$$

The quantity l_1 is determined when solving the linearized system of equations (see Section 5). The linearization is carried out with respect to a one-wave periodic solution which depends on a single parameter C (it is possible that C = var, V = const). This is a possible alternative approach to finding a two-wave structure according to the principle used in Section 6.

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